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Voting matters

for the technical issues of STV

The Electoral Reform Society

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Editorial

On the 17th May 2002, the Dáil constituencies of Meath, Dublin North and Dublin West used an experimental system for electronic voting. It is expected that this system will be used exclusively for local and national elections in the Irish Republic in the near future.

Of course, the three constituencies used the same electoral rules as in the other 40 — essentially a hand-counting system which has at least one ‘problem’ in that the result can depend upon the order in which papers are transferred on a surplus. Examination of such issues has traditionally always been hampered by the lack of complete information of all the preferences expressed by the voters.

We now have a significant step forward for electoral studies since the Irish electronic voting results includes the complete data input to the electronic counting software. One can reasonably expect future issues of *Voting matters* to analyse this data.

The first paper in this issue is indeed an analysis of Irish election data, but only uses the result sheets. Philip Kestelman shows statistically significant bias according to the alphabetic position (on the ballot paper). I might add that even a casual inspection of the full data mentioned above shows a tendency for the final few preferences to be in strictly ascending or descending order.

In the second paper, Eivind Stensholt considers the problem when additional support for a candidate results in that otherwise elected candidate not being elected. This property of *non-monotonicity* applies even to the case of electing a single candidate, as shown in this paper. On the other hand, the paper indicates that it is relatively rare.

In the third paper, Markus Schulze considers an algorithm for electing candidates with preference voting proposed by Professor Sir Michael Dummett. Sir Michael has chosen not to respond to the criticisms made.

In the last paper, David Hill and Simon Gazeley produce a new STV-like algorithm which merges the ideas of Condorcet and STV. The advantage of this algorithm is to avoid the property of all conventional STV algorithms of premature exclusion, such as for a universal second-choice candidate. On the other hand, this method has the disadvantage of later preferences could possibly upset earlier ones in rare cases.

McDougall Trust: STV Resources CD

A proof copy was prepared in February, but the publication date has not yet been agreed.

Brian Wichmann.

Positional Voting Bias Revisited

Philip Kestelman

Introduction

It is widely supposed that candidates appearing high on ballot-forms enjoy a considerable electoral advantage. In a highly influential paper on the 1973 General Election to the Irish Dáil, by multi-member Single Transferable Voting (STV), Robson and Walsh (1974) observed that Deputies (TDs) over-represented candidates with A-C surnames. Compared to randomly sampled Irish electors, “The under-representation of M-O names among politicians is very striking”.

Proportionality conventionally measures the relationship between numbers of Party *votes* and seats (regardless of candidates). Despite a probable age bias, we are hardly concerned that seats considerably over-represent first preferences for *incumbent* candidates; let alone that incumbents are far more likely to be elected than ‘excumbents’ (non-incumbents).

On the other hand, we are concerned not only that seats should proportionally represent votes for *women* candidates, but also that seats should be proportional to women *candidates*, in the interests of Parliament representing society. In respect of ballot-form *position*, we are primarily concerned with the relationship between numbers of *candidates* and seats (regardless of votes), by surname initial, when candidates are listed surname-alphabetically on ballot-forms.

Electability

This article mainly evaluates positional voting bias in the last 12 general elections in the Irish Republic (1961-97). Electability is quantified in terms of an Electability Index (S%/C%): the ratio of a seat-fraction (S%) to a candidate-fraction (C%); and of a Relative Electability Ratio: the ratio between specified Electability Indices.

Aggregating all 12 elections (Total S/C = 1,875/4,594), Upper/Lower half surname A-J/K-Z Electability Indices were 1.11/0.88, with a statistically highly significant Relative Electability Ratio of 1.26 (P<0.001). By comparison, alphabetically Upper/Lower half forename A–L/M–Z Electability Indices were 1.01/0.99, with an insignificant Relative Electability Ratio of 1.01 (P > 0.05).

Cumbency

In 1961-97, most incumbent candidates (S/C = 1,404/1,687 = 83 percent) were re-elected; whereas few excumbents (471/2,907 = 16 percent) were elected, rendering them more susceptible to alphabetic disproportionality. Surname A-J/K-Z Electability Indices (S%/C%) were 1.01/0.98 for incumbents, and 1.15/0.86 for excumbents, with Relative Electability Ratios of 1.03 (P>0.05) and 1.34 (P<0.05), respectively.

The last 12 Irish general elections have consistently over-represented excumbent candidates with A-C surnames; under-representing those with K-M surnames (overall S%/C%, 1.27 and 0.81: Table A). Even combining the 12 elections into three quartets leaves considerable variability in both forename and surname Electability Indices.

Table A: *Excumbent* Electability Index, by Elections and Forename/Surname initial letter: Irish Republic, 1961-97 (12 general elections: Dáil Éireann, 1962-98).

Name		Elections: Electability Index (S% / C%)			
Initial letter		1961-97	1961-73	1977-82	1987-97
Forename	A-F	1.04	1.34	1.11	0.88
	G-L	1.08	1.08	0.98	1.16
	M-P	1.02	0.82	1.14	1.08
	Q-Z	0.82	0.84	0.72	0.85
Ratio (A-L/M-Z)		1.14	1.42*	1.10	1.01
Surname	A-C	1.27	1.38	1.19	1.24
	D-J	1.04	1.04	1.06	1.04
	K-M	0.81	0.73	0.84	0.83
	N-Z	0.92	0.91	0.91	0.94
Ratio (A-J/K-Z)		1.34*	1.47*	1.29	1.27

* P < 0.05

In 1961-73, excumbent forename and surname alphabetic biases were equally convincing (P<0.05); but insignificant subsequently. Ironically in 1973, the Relative Electability Ratio for A-L / M-Z forenames (2.76) exceeded that for A-J / K-Z surnames (1.57)! The pitfalls of generalising from a single election are manifest.

District Magnitude

Surname disproportionality was virtually confined to four- and five-member STV constituencies: only three-member constituencies returned TDs more-or-less faithfully reflecting excumbent surnames (Table B). Magnitude-specific surname A-J/K-Z Relative Electability Ratios proved statistically insignificant, but much closer to unity in three-member constituencies (1.25, 0.89 and 0.72) than in four-member constituencies (1.62, 1.36 and 1.51), or in five-member constituencies (2.05, 1.77 and 1.42), in 1961–73, 1977–82 and 1987–97, respectively.

Table B: *Excumbent* Electability Index, by District Magnitude (seats per constituency) and Surname initial letter: Irish Republic, 1961-97 (12 general elections: Dáil Éireann, 1962-98).

Surname Initial letter	Magnitude: Electability Index (S% / C%) Total+	3	4	5
A-C	1.27	1.10	1.51	1.21
D-J	1.04	0.87	0.98	1.28
K-M	0.81	1.05	0.66	0.73
N-Z	0.92	0.98	0.96	0.79
Ratio (A-J/K-Z)	1.34*	0.96	1.51	1.64

*P < 0.05 +Including a few two-member constituencies.

District Canditude and Position

Interestingly, the 1961-97 aggregate, excumbent Relative Electability Ratio by surname (A-J/K-Z) proved identical with that by ballot-form position (Upper/Lower = 1.34: P<0.05). Like the surname A-J/K-Z Relative Electability Ratio with district magnitude (the number of seats per constituency), the positional Upper/Lower Relative Electability Ratio increased with district ‘canditude’ (the number of candidates per constituency: Table C).

Table C: *Excumbent* Electability Index, by District Canditude (candidates per constituency) and Ballot-form Position: Irish Republic, 1961-97 (12 general elections: Dáil Éireann, 1962-98).

Ballot-form Position+	Canditude: Electability Index (S% / C%) Total	4-8	9-11	12-21
Top	1.30	1.22	1.26	1.41
Upper-middle	0.98	0.86	1.06	1.01
Lower-middle	0.90	1.15	0.91	0.76
Bottom	0.83	0.83	0.81	0.86
Ratio (Upper/Lower)	1.34*	1.13	1.37	1.49

* P < 0.05 +Excluding odd-Canditude mid-candidates.

Party Policy

Both main political parties in the Irish Republic (Fianna Fáil and Fine Gael) have staunchly denied over-nominating candidates appearing high on ballot-forms[9]. Table D analyses the surname-alphabetic distribution of FF and FG excumbent candidates, compared with other (non-FF + FG) excumbents, in terms of a Relative Nomination Index, over time.

Table D: Two Main Party *Excumbent* Relative Nomination Index, by Elections and Surname initial letter: Irish Republic, 1961-97 (12 general elections: Dáil Éireann, 1962-98).

Surname Initial letter	Elections: Relative Nomination Index (Fianna Fáil +Fine Gael C%/ Other C%) 1961-97	1961-73	1977-82	1987-97
A-C	1.46	1.07	1.63	1.57
D-J	0.96	1.02	1.13	0.84
K-M	0.92	1.09	0.79	0.87
N-Z	0.81	0.85	0.69	0.92
Ratio (A-J/K-Z)	1.35***	1.08	1.82***	1.25

*** P < 0.001

Evidently since 1977, both main parties have greatly over-nominated A-C surname candidates (and/or other parties have under-nominated them); with the honours evenly divided between Fianna Fáil and Fine Gael. Relative to the publication of Robson and Walsh (1974), the timing may not have been entirely coincidental!

Electorate

Robson and Walsh (1974) observed that the alphabetic distribution of surname initial letters differed insignificantly between randomly sampled Irish electors and excumbent candidates at the 1973 Irish General Election. Presumably nowadays, the surname initials of *electors* are rather better represented by excumbent, non-FF + FG candidates; and Table E compares overall seat-fractions, by surname initial, with excumbent, non-FF + FG candidate-fractions.

This Surname Concentration Index (Total S%/Excumbent, non-FF + FG C%) highlights Dáil Éireann over-representing A-C surnames in the Irish electorate; while under-representing K-M surnames. Despite the lower surname A-J/K-Z Concentration Ratio since 1987 (1.35), A-C surname electors remain over twice as likely as K-M surname electors to become TDs.

Table E: Surname Concentration Index, by Elections and Surname initial letter: Irish Republic, 1961-97

(12 general elections: Dáil Éireann, 1962-98).

Surname Initial letter	Elections: Surname Concentration Index (Total S% /Excumbent, non-Fianna Fáil +Fine Gael C%) 1961-97	1961-73	1977-82	1987-97
A-C	1.65	1.56	1.66	1.58
D-J	1.01	1.13	1.13	0.91
K-M	0.70	0.76	0.63	0.73
N-Z	0.85	0.68	0.85	0.99
Ratio (A-J/K-Z)	1.65***	1.85***	1.86***	1.35**

** P < 0.01 *** P < 0.001

Other STV Elections

Compared to the last 12 Irish general elections, with a total surname A-J/K-Z Relative Electability Ratio of 1.26 (P<0.001), the last five European elections in the Irish Republic (1979-99: Total S/C = 75/234) have yielded a higher but statistically insignificant surname A-J/K-Z Relative Electability Ratio of 1.37 (P > 0.05) [5].

On the other hand, the last five Irish Local Elections (1979–99: Total S/C = 4,918/10,250) disclosed a lower surname A-J/K-Z Relative Electability Ratio of 1.12 (P<0.001)[3]. Perhaps better acquainted with local government candidates, voters discriminate more individually; numbering their preferences regardless of alphabetical order.

At the 1973 Assembly Election in Northern Ireland, Robson and Walsh (1974) attributed eight out of 78 Seats to positional voting bias. Yet at the 1998 Northern Ireland Assembly Election (Total S/C = 108/296), the surname A-J/K-Z Relative Electability Ratio fell below unity (0.87: P>0.05)[4]. Certainly, parties are more sharply differentiated in Northern Ireland than in the Irish Republic.

Discussion

Using forenames as controls, surname-alphabetic electability valuably measures voters' lack of discrimination between candidates within parties. Neither voters nor the Irish electoral system (STV) can be reproached for any positional voting bias.

However, Dáil Éireann remains surname-alphabetocratic, over-representing candidates with A-C surnames, while under-representing excumbents (non-incumbents) with K-M surnames (Table A: compare Table E): especially in constituencies with over three seats (Table B), and/or over eight candidates (Table C).

Perhaps aware of Robson and Walsh (1974), Ireland's two main parties (Fianna Fáil and Fine Gael) have apparently over-nominated A-C excumbents (notably since 1977: Table D). However, thus acting on the belief of increased electability may itself increase A-C surname over-representation: aggregating the last 12 Irish general elections (1961-97), excumbent S%/C% for FF + FG (1.67) was considerably higher than for other parties (0.43).

Reassuringly, aggregating all 12 general elections (1961-1997), the excumbent Surname Relative Electability Ratio (S/C ratio: A-J/K-Z=1.34 overall) proved significantly higher for FF+FG (1.28: P<0.05) than for the other candidates (1.03: P>0.05). however, it remains unclear whether the two main parties have benefited from A-C over-nomination.

Darcy and McAllister (1990) found "no evidence for position advantage for political parties in any election". Their review concluded that positional voting bias may be eliminated by removing its causes: notably, compulsory voting; completion of all preferences; and ballot-forms not indicating candidates' Party affiliation (as in Ireland before 1965[7]).

On the strength of the 1973 Irish General Election, Robson and Walsh (1974) advocated randomising the order of candidates on ballot-forms. Citing Robson and Walsh (1974), Sinnott[8] suggested that the problem could "easily be eliminated by arranging the names in a number of different randomised orders on different sets of ballot papers".

At the Dublin High Court in 1986, Mr Justice Murphy accepted that candidates with surname initials high in the alphabet were over-represented but, noting that alphabetic order helped voters to find candidates, he found it constitutional[9]. Indeed, the voter's predicament is paramount; and to avoid the palpable frustrations of randomised ballot-forms in locating preferred candidates, a reasonable compromise might be to print half the ballot-forms in surname-alphabetic order, with the other half in the reverse order — if positional voting bias really matters.

Acknowledgement

I am indebted to David Hill for statistical advice. Statistical significance was calculated by combining election-specific, one-tailed exact two-by-two table probabilities[6].

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Nonmonotonicity in AV

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Introduction

Nonmonotonicity arises with STV when apparent additional support for a candidate, A, at the expense of another candidate, C, causes a third candidate, B, to be elected. Without the additional support, A would be elected. Thus the additional support actually costs A the election. This unfortunate property in the standard variations of STV is linked to the elimination of candidates in the counting process[6], and it is unavoidable unless some compromise is made with the principle that a voter's later preferences cannot influence the fate of the voter's earlier preferences.

How frequently will it happen that a candidate is not elected, but might have been elected if some of his or her support had gone to another candidate instead? That depends on the voters' behaviour. Based on standard assumptions on the distribution of voter preference, modified by empirical evidence of voter behaviour, the frequency is estimated for elections in which 1 candidate is elected from 3. This is the Alternative Vote (AV), a single-seat version of STV.

It is also shown how the nonmonotonicity is related to the Condorcet paradox in which one majority prefers B to A, another majority prefers A to C, and a third majority prefers C to B. In all elections considered, each voter is assumed to give a complete preference list.

For example, consider an election (from a simulation with 10000 voters) with

```
 475        ABC
3719        ACB
 390        CAB
2110        CBA
  41        BCA
3265        BAC
```

No candidate has 50% of the first preference votes. C, with only 2500 first preference votes is eliminated, and finally B defeats A with 5416 votes to 4584. However, if x of the ACB-voters vote "strategically" CAB instead, the election may turn out differently. Then the profile is

```
 475        ABC
3719−x      ACB
 390+x      CAB
2110        CBA
  41        BCA
3265        BAC
```

If $x > 806$, C with 2500+x overtakes B with 3306, and if x<888, A is still ahead of B with 4194−x to 3306. Thus, with $806 < x < 888$, B gets eliminated, and finally A defeats C with 7459−x votes to 2541+x.

The example also shows the Condorcet paradox of cyclic majorities. In pair-wise encounters A defeats C with 7459−x to 2541+x, C defeats B with 6219 votes to 3781, and B defeats A with 5416 votes to 4584. However, in real elections with 3 candidates cyclic majorities become very rare as the number of voters increases. One indicator of unrealism is that the cyclic order ABCA receives only $475+390+41+x = 906+x$ votes while ACBA receives $3719+2110+3265-x = 9094-x$ votes. In real elections the votes are distributed in the 6 categories in a more harmonious way.

If nonmonotonicity occurs in a real election, the scenario is most likely that there is a plurality winner, A (with the largest number of first preference votes), another Condorcet winner, B (who defeats each other candidate in pair-wise encounters), and a third candidate, C (who is last in first preference votes). Such an example, from the same simulation, is

```
2996        ABC
1122        ACB
 875        CAB
2046        CBA
1431        BCA
1530        BAC
```

Here C is eliminated and B wins the AV-election. If x voters switch from ACB to CAB, and $40 < x < 648$, then B is eliminated and A wins. It turns out that if AV is modified and A declared winner in the few cases like this, nonmonotonicity is eliminated. Instead, however, another principle will be violated: B may win by a suitable vote transfer from BAC to BCA.

3-candidate elections may be classified according to how well the "electoral cake model" in Stensholt[5] may be fitted; the figure on page 7 shows a good fit. The model may be fitted quite well to most real elections. When simulated elections are classified, election P is considered more "realistic" than election Q if the model fits P better than Q. When better fit, i.e. more "realism", is demanded, the frequency of the

Condorcet paradox will approach 0. Nonmonotonicity, however, occurs in about 0.90% of all simulated "realistic" elections. Two real elections (37 candidates, 63 voters) and (14 candidates, 115 voters) have been checked, with nonmonotonicity in, respectively, 0.66% and 1.10% of the candidate triples.

A description of nonmonotonicity by means of inequalities

A possible preference distribution P in an election with 3 candidates, A, B, and C (a *profile* in the social choice vernacular), consists of a sequence of 6 non-negative numbers.

P = $(p\ \ q\ \ r\ \ s\ \ t\ \ u)$,

These are the numbers (absolute or relative) of voters with preference ranking respectively: ABC, ACB, CAB, CBA, BCA, BAC.

If x of the ABC-voters and y of the ACB-voters change to vote CAB, there is a new profile Q:

Q = $(p-x\ \ q-y\ \ r+x+y\ \ s\ \ t\ \ u)$.

Nonmonotonicity occurs if B is AV-winner in P and A in Q despite the natural expectation that the candidate A is weaker in Q than in P. The story is told in 9 inequalities.

$$r+s+t+u > p+q \quad (1)$$
$$p+q+r+s > t+u \quad (2)$$
$$p+q > r+s \quad (3)$$
$$t+u > r+s \quad (4)$$
$$s+t+u > p+q+r \quad (5)$$
$$p+q+t+u-(x+y) > r+s+(x+y) \quad (6)$$
$$r+s+(x+y) > t+u \quad (7)$$
$$p+q-(x+y) > t+u \quad (8)$$
$$u+p+q-(x+y) > r+s+t+(x+y) \quad (9)$$

A translation to non-mathematical language links the inequalities to the AV rules. (1, 2): In P, neither A nor B have 50% of the first preference votes. (3, 4): In P, C has the lowest number of first preference votes. (5): In P, B wins over A (after elimination of C). (6): In Q, C does not reach 50% first preference votes. (7): In Q, C passes B in first preference votes. (8): In Q, A keeps more first preference votes than B. (9): In Q, A wins over C (after elimination of B).

However, the mathematical version (1-9) is easier to analyse. Write (7, 8, 9) equivalently as

$$\min[p+q-t-u, (u+p+q-r-s-t)/2] > x+y > t+u-r-s \quad (10)$$

Thus numbers x and y satisfying (7, 8, 9) exist if and only if (11) and (12) hold:

$$p+q+r+s > 2t+2u \quad (11)$$
$$p+q+r+s > 3t+u \quad (12)$$

Moreover, (1), (2), (3), and (6) are redundant because of (5), (11), (4 and 8), and (9), respectively. Therefore the $p+q$ supporters of candidate A can turn defeat in P to victory in Q if and only if (4, 5, 11, 12) all hold. Then $x+y$ of them vote "strategically" CAB, with $x+y$ as in (10).

A profile where a candidate may be helped by being ranked lower in some ballots without any other change in any ballot will be called a nonmonotonic profile for the election method considered. In discussing various election rules, it is also useful to have an "absolute" definition: A profile is then nonmonotonic if it is so for AV. A monotonic election method is one without nonmonotonic profiles. AV, and the usual STV-variations are nonmonotonic because of the elimination rules. By the criteria (4, 5, 11, 12)

$$p+q > t+u > r+s \quad (13)$$

Thus, in P, A is plurality winner (first past the post), while B beats A and A beats C in pair-wise comparisons by (5) and (9). This we will call nonmonotonicity of type ABC. There are six types of nonmonotonic profiles: ABC, ACB, CAB, CBA, BCA, and BAC.

Connection to the Condorcet paradox; a geometric description

The Condorcet paradox occurs together with ABC-type nonmonotonicity when also C beats B in pair-wise comparison, i.e.

$$q+r+s > t+u+p \quad (14)$$

Otherwise B is the Condorcet winner, i.e. B defeats each opponent in a pair-wise contest. The strategic voting of the $x+y$ voters who honestly support A is then designed to take the AV victory away from Condorcet winner B to plurality winner A. Define E, F, G, H, K as functions of the profile:

$$E=-r-s+t+u$$
$$F=-p-q-r+s+t+u$$
$$G=p+q+r+s-2t-2u \quad (15)$$
$$H=p+q+r+s-3t-u$$
$$K=-p+q+r+s-t-u$$

When all possible profiles are standardized, e.g. to $p+q+r+s+t+u=12$, as in the table below, they form a 5-dimensional simplex with 6 corners — a higher dimensional analogue of the familiar 3-dimensional simplex (tetrahedron) with 4 corners and 4 triangular sides.

Profile	*p*	*q*	*r*	*s*	*t*	*u*	E	F	G	H	K	100ε
P01	4	0	2	2	2	2	0	0	0	0	–4	5.61
P02	4	0	2	2	0	4	0	0	0	4	–4	0.00
P03	4	0	0	4	2	2	0	4	0	0	–4	0.00
P04	4	0	0	4	0	4	0	4	0	4	–4	0.00
P05	6	0	0	3	3	0	0	0	3	0	–6	0.00
P06	6	0	0	3	0	3	0	0	3	6	–6	0.00
P07	6	0	0	2	2	2	2	0	0	0	–8	0.00
P08	6	0	0	2	0	4	2	0	0	4	–8	0.00
P09	0	4	2	2	2	2	0	0	0	0	4	5.61
P10	0	4	2	2	0	4	0	0	0	4	4	20.69
P11	0	4	0	4	2	2	0	4	0	0	4	20.69
P12	0	4	0	4	0	4	0	4	0	4	4	41.35
P13	0	6	0	3	3	0	0	0	3	0	6	32.54
P14	0	6	0	3	0	3	0	0	3	6	6	39.77
P15	0	6	0	2	2	2	2	0	0	0	4	17.27
P16	0	6	0	2	0	4	2	0	0	4	4	38.72

By (4, 5, 11, 12) the nonmonotonic profiles of ABC-type form a convex subset S of this simplex, given by

$$E > 0, F > 0, G > 0, H > 0 \qquad (16)$$

The Condorcet paradox occurs if K > 0 too. The profiles in the table are the corners of the closure of S and have non-negative E, F, G, H.

In the right hand column, ε = ε(P) is a continuous function of the profile P, defined in Stensholt[5]. By its definition, $0 < \varepsilon < 3\sqrt{3}/4\pi \approx 0.4135$. Generally ε is well below 0.01 in profiles from real elections with many voters. Any profile P satisfying (16) may be written as

$$P = k01\cdot P01 + k02\cdot P02 + k03\cdot P03 + ... + k16\cdot P16 \qquad (17)$$

with non-negative k_j and k01 + k02 + k03 +... + kl6 = 1.

To a profile P = (*p q r s t u*) we may assign a twin profile P* = (*q p r s t u*). Thus P**=P and $P_i^* = P_{i+8}$, i=1, 2,.., 8. If P is a nonmonotonicity profile of type ABC, so is P*. With P as in (17), then

$$P^*=k09\cdot P01+k10\cdot P02+..+k16\cdot P08+ k01\cdot P09+k02\cdot P10+..+k08\cdot P16, \quad (18)$$

$$K(0.5\cdot[P + P^*]) = 0.5\cdot[K(P) + K(P^*)] = -2\cdot(k07 + k08 + k15 + k16) \leq 0 \quad (19)$$

Thus the profile 0.5 [P + P*], midway between P and P*, will never give the Condorcet paradox, but it is on the borderline if and only if k07 = k08 = kl5 = kl6 = 0 Somewhere between 1/3 and 2/3 along the line segment from P to P*, K = 0. From the K-column in Table 1 it is clear that, with many voters, somewhere between 33% and 50% of all nonmonotonicity profiles also have a Condorcet cycle. However, they are not all equally likely to occur in real elections.

Simulation and reality

One million random 3-candidate profiles were generated with uniform probability in the simplex. The distribution is known as the *Impartial Anonymous Culture* (IAC). The IAC also depends on the number of voters, but the simulation corresponds to the limit case of infinitely many voters. Actually about 100 voters would give quite similar results.

In 3621 of the IAC-generated profiles were E>0, F>0, G>0, H>0. As there are six nonmonotonicity types, about 6×0.3621% ≈ 2.17% of the profiles are nonmonotonic. Among these 3621, 1602, i.e. ≈ 44.24% also had K>0, indicating a Condorcet cycle in the profile: A beats C beats B beats A. For comparison, 6.25% of all IAC-profiles have a Condorcet cycle [2,5].

In real elections the cycle frequency is much lower. That is due to a structure in the profiles, which may come from the voters having some common perception of the "political landscape" although they have placed themselves in different positions and rank the candidates accordingly [5].

Imagine that the voters are distributed with uniform density in a circular disc, that candidates A, B and C are among them, and that a voter ranks the candidates according to their distance from the voter's position. In a pair-wise comparison between A and B, B wins if and only if B is closer than A to the circle centre. A and B divide the voters between them with the mid-normal to the line segment AB as dividing line. Similarly the mid-normals for BC and AC divide the disc. The three candidates split the "voter cake" in six pieces by three straight cuts through one common point, each piece getting an area proportional to the number of votes with the corresponding ranking of the candidates. In a model like this, the Condorcet paradox can never occur except in a degenerate form with all cuts through the circle center, and *p*=*s*, *q*=*t*, *r*=*u*.

Empirically, the electoral cake model fits reasonably well for 3-candidate profiles from real elections with a large number of voters. That is why the Condorcet paradox is rare. The function ε(P) measures the deviation of P from the model. For the examples in the introduction,

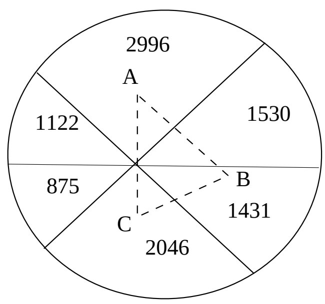

Figure giving the profile (2996 1122 0875 2046 1431 1530) which fits well with the "electoral cake" model.

ε(0475, 3719, 0390, 2110, 0041, 3265) = .174035768

ε(2996, 1122, 0875, 2046, 1431, 1530) = .000000108 (see figure).

Among the simulation profiles with small ε(P), about 0.15% were nonmonotonic of ABC-type. This suggests an estimate of 0.90% for the probability for nonmonotonicity in a candidate triple in real elections with many voters.

In an election with 63 voters and 37 candidates at the author's institution, 51 of the 37×36×35/6 = 7770 triples were nonmonotonic, a fraction of 0.66%. In these 51 triples, the Condorcet paradox occurred only 7 times, i.e. much less than the 44.24% in the full IAC-simulation. In another election in the same place, with 115 voters and 14 candidates there were 4 nonmonotonicity triples out of 14×13×12/6 = 364, i.e. 1.10% and the Condorcet paradox occurred in none of them. Comparison with the simulation requires some caution since the triple profiles in an election with many candidates cannot be assumed stochastically independent.

Conclusion

In an election with 3 candidates, A, B and C, let A be plurality winner. In the vast majority of elections, there will also be a unique Condorcet winner. If A also happens to be Condorcet winner, A wins the AV-election. That cannot be very controversial.

So assume B is Condorcet winner, which means that B wins if A or C is eliminated. B may win with very few first preference votes in the ballots, but electing B means that there are no "wasted votes". The "plurality ideology" may also be modified to avoid wasting votes by eliminating B; then the supporters of B are allowed to influence the choice between A and C. An election method that *always* eliminates a Condorcet winner who is not also a plurality winner, may seem strange. However, it would, arguably, be a democratic improvement of the plurality method that is in wide use today. It preserves the "plurality ideology" as well as possible, preferring to let centre voters decide between "right" and "left" rather than filling an assembly with centre politicians.

AV can be seen as a compromise between the "plurality ideology" and the "Condorcet ideology". There are two possibilities.

(I) If B has the smallest support in terms of first preference votes, i.e. $p+q > r+s > t+u$, then B is eliminated.

(II) If B is number 2 in terms of first preference votes, i.e. $p+q > t+u > r+s$, then B is the AV-winner.

Nonmonotonicity occurs in (II) if A has a number of surplus first preference votes that could be transferred to C in a way that benefits A. Such transfer is not a part of AV, but this can be remedied in the spirit of STV if the transfer rule is extended. When (16) holds, let the necessary number of surplus votes be transferred from voter categories ABC and ACB to CAB if this lets C become number 2 in terms of first preference votes, and still lets A win against C after elimination of B. This transfer of first preference votes from A to C involves only voters who prefer A to B (categories ABC, ACB, CAB), and it may be implemented in the counting process when it helps A to win instead of B.

An obvious argument against such a procedure is that it occasionally may violate the cherished principle that my second preference should never hurt my first preference. To see this, consider first standard AV. Then C is eliminated after examination of first preferences only. The second preferences of C's supporters become available, and either A (plurality winner) wins or B (Condorcet winner if one exists) wins. Among the conditions in (16) for an extra transfer of votes from A to C, the three first only involve first preference votes: $p+q$, $r+s$, $t+u$. The inequality H > 0 requires information about t and u, i.e. about the second preferences of B's supporters. This allows for strategic voting on behalf of B. Let z voters move from BAC to BCA. Then according to (15) the requirement H > 0 is sharpened to

$$p+q+r+s-3t-u-2z>0.$$

The strategy is to break this condition, which is achieved if and only if

$$p+q+r+s-3t-u \le 2z \le 2u$$

Such strategy is possible if and only if

$$p+q+r+s+t+u \le 4(t+u),$$

i.e. if and only if B has at least 25% of the first preference votes. This will, however, always be the case when the extra transfer rule is invoked, because by (1) A has less than 50% of the first preference votes and by (4) B has more first preference votes than C. AV with extra transfer violates the principle exactly when standard AV violates monotonicity.

In 3-candidate elections, voters may be offered one of two guarantees:

1) You can never hurt a candidate by an upwards move;

2) You can never hurt a candidate by a change in the subsequent ranking.

In about 99% of the elections, the profile is monotonic. Then AV and AV with extra transfer satisfy both 1) and 2), as no extra transfer is done. In the remaining cases, standard

AV picks the Condorcet winner and violates 1) but not 2), while AV with extra transfer picks the plurality winner and violates 2) but not 1). Which of the two guarantees is then most important?

With more candidates, it becomes more complicated to study nonmonotonicity in AV. With 5 candidates, A, B, C, D, and E, there are 10 triples, and each candidate takes part in 6 triples:

{A,B,C}, {A,B,D}, {A,B,E}, {A,C,D}, {A,C,E}, {A,D,E}, {B,C,D}, {B,C,E}, {B,D,E}, {C,D,E}.

After all but 3 candidates are eliminated, there is a final triple, say {A,B,C}. If AV is adopted in more than 600 constituencies, as in a Westminster election, there will generally be some with nonmonotonicity in {A,B,C}. How bad will criticism from frustrated supporters of a non-elected plurality winner in such cases be for people's trust in standard AV?

If A, B and C are much stronger than all other candidates, it may be enough to implement the extra vote transfer in {A,B,C} in order to cope with most nonmonotonic profiles. Nonmonotonicity is reduced, at a price: How bad will criticism from frustrated supporters of a non-elected Condorcet winner in such cases be for people's trust in AV with extra transfer?

The purpose of elimination is to find the opponent for A in the final pair, so B or C must be eliminated. The extended transfer rule only adjusts the border between elimination of B and elimination of C. Is an election of B due to honest first priority from A's supporters more tolerable than election of A due to honest subsequent ranking from B's supporters?

Can we achieve monotonicity with more than 3 candidates, at a reasonable price? Perhaps a recursive idea may work. Assume that the set of profiles S with *n* candidates has been subdivided into *n* subsets $S = S_1 \cup S_2 \cup ... \cup S_n$, so that candidate *i* wins with profile in S_i and that this election method is monotonic. With *n*+1 candidates left, eliminate Z with the lowest number of first preference votes. If that leads to a profile in S_Y and $X \neq Y$, then allow an extra transfer of first votes from X to Z or even to more candidates in order to eliminate another candidate and obtain an *n*-candidate profile *i* S_x. The possibility of saving more candidates than Z from elimination by an extra transfer raises the question of whether X is uniquely defined.

A more radical measure is to count in each triple separately, implementing the extra transfer. "Triple-AV" then gives a candidate one point for a triple victory, and achieves monotonicity. It is similar to Copeland's method[1,3,4], which gives one point for each victory in a pair-wise comparison and avoids Condorcet cycles. On the other hand, the price for monotonicity with triple-AV may well be too high in terms of violations of the principle.

An axiomatic study of election theory reveals some basic impossibilities. Certain combinations of nice properties cannot be realized simultaneously in one election method. To achieve monotonicity, one must sacrifice the principle. On the other hand, only in the few cases where (16) holds, will triple-AV find another triple winner than standard AV.

Three papers in *Voting matters*[6,7,8] deal with nonmonotonicity and related problems. One theme is the axiomatic understanding of election methods: which combinations of desirable properties are theoretically incompatible? That kind of knowledge is important for everyone concerned with "how to choose how to choose". An axiomatic approach, however, needs a clearly formulated and manageable conceptual frame. As part of this frame, it must be clearly stated what kind of preference relations the voters are allowed to express. One may restrict ballots to be complete, or to conform to a linear listing of the alternatives (single-peak condition), etc. Within this frame the axiomatic investigator must take into account all possible profiles without any extra screening against unrealistic profiles. Even a highly concocted profile may be a counter-example that kills a hypothesis; lack of realism is no objection if the profile formally is within the axiomatic frame. According to Stensholt[5] a bound on the function ε(P) of the 3-candidate profile P is useful to screen off most of the unrealistic profiles generated in a simulation. However, a criterion like $\varepsilon(P) < 0.01$ does not seem suitable for axiomatic treatment. Axiomatics must be followed up by other approaches, e.g. comparisons of election methods on simulated and empirical data.

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On Dummett's "Quota Borda System"

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In two books[1,2], in his submission to the Jenkins Commission[3], and at a number of conferences, Michael Dummett has promoted a preferential voting method where one successively searches for solid coalitions of increasing numbers of candidates and where, when one has found such a solid coalition, one declares the candidates with the best Borda scores elected. Dummett calls his method "Quota Preference Score" (QPS) or "Quota Borda System" (QBS). He writes that his method "has never been in use, but was voted the best at a conference on electoral reform held in Belfast with representatives of all parties" [3]. In his book *Voting Procedures*, he describes this method as follows (where *v* is the number of voters, *S* is the number of seats, **C** is the number of candidates, and the "preference score" is the Borda score) [1, pp. 284-286]:

The assessment will proceed by stages, all but the last of which may be called "qualifying stages": it will of course terminate as soon as all *S* seats have been filled. We may first describe the assessment process for the case when *S* is 2 or 3. At stage 1, the tellers will determine whether there are any candidates listed first by more than 1/(*S*+1) of the total number *v* of voters: if so, they immediately qualify for election. If seats remain to be filled, the preference scores of all candidates not qualifying at stage 1 will then be calculated. At stage 2, the ballot papers will be scrutinized to see if there is any pair of candidates, neither of whom qualified at stage 1, to whom more than *v*/(*S*+1) voters are solidly committed: if so, that member of the pair with the higher preference score now qualifies for election. If seats remain to be filled, the tellers will proceed to stage 3, at which they will consider sets of three candidates, none of whom has already qualified. If more than *v*/(*S*+1) voters are solidly committed to any such trio, that one with the highest preference score qualifies for election. In general, at the qualifying stage *i*, the tellers determine whether, for any set of *i* candidates none of whom has so far qualified, there are more than *v*/(*S*+1) voters solidly committed to those candidates; if so, the member of the set with the highest preference score qualifies for election at stage *i*. If there still remain seats to be filled after all the qualifying stages have been completed, they will be filled at the final stage by those candidates having the highest preference scores out of those who have not yet qualified. (. . .)

When *S* = 4, however, it may be thought that a body of voters, amounting to more than two-fifths of the electorate and solidly committed to two or more candidates, is entitled to 2 of the 4 seats. To achieve this, the assessment process must be made a little more complex. Stage 1 will proceed as before, and, at stage 2, the same operation must be carried out as described above. Before proceeding to stage 3, however, the tellers must also consider every pair of candidates of whom one qualified at stage 1 and the other did not: if more than 2·*v*/(*S*+1) voters are solidly committed to such a pair, that one who did not qualify at stage 1 qualifies at stage 2. (Note that, if more than 2·*v*/(*S*+1) voters are solidly committed to two candidates, one of them must qualify at stage 1.) Likewise, at each qualifying stage *i*, the tellers must ask, of every set of *i* candidates of whom at most one has already qualified, whether more than 2·*v*/(*S*+1) voters are solidly committed to those candidates. If so, and none of them has previously qualified, the two with the highest preference scores will now qualify; if one of them qualified at an earlier stage, that one, of the rest, who has the highest preference score will qualify at stage *i*. (. . .)

In general, at stage *i*, the tellers must ask, of each set of voters solidly committed to i candidates, what multiple of *v*/(*S*+1) members it contains, up to *i*·*v*/(*S*+1). If it contains more than *v*/(*S*+1) voters, at least one of the *i* candidates will qualify for election; if it contains more than 2·*v*/(*S*+1), at least two will qualify; if 3 ≤ *i* and it contains more than 3·*v*/(*S*+1), at least three will; and so on, up to the case in which it contains more than *i*·*v*/(*S*+1) voters, when all *i* candidates will qualify.

This description of QBS seems unnecessarily long. Usually, Dummett offers a significantly shorter description. For example, in his submission to the Jenkins Commission he writes[3]:

The scruntineers can first mark as elected any candidate ranked highest by a sufficiently large minority (one-sixth of the voters in a five-member constituency, etc.). Then, having calculated the Borda counts of all remaining candidates, they can discover whether any set of from two to five candidates receives solid support from a sufficiently large minority: if so, that candidate in the set with the highest Borda count is marked as to be elected. The remaining seats will be filled by the candidates most generally acceptable to the electorate as a whole, i.e. those with the highest Borda counts.

In my opinion, a problem of the shorter description is that readers could mistakenly believe that the order in which the solid coalitions are considered at each stage and the question at which stages the different candidates have qualified were unimportant. However, example 1 demonstrates that they are decisive.

Example 1 (v = 100; S = 2; **C** = 5):

```
29 DBCEA.
17 ABDCE.
17 BADCE.
17 CADBE.
13 ACDBE.
 7 CABDE.
```

The Borda scores are 243 for candidate A, 250 for candidate B, 227 for candidate C, 251 for candidate D, and 29 for candidate E. Table 1 lists all solid coalitions. At stage 1, no candidate qualifies for election. At stage 2, it is observed that more than $v/(S+1)$ voters are solidly committed to the candidates A and B and that more than $v/(S+1)$ voters are solidly committed to the candidates A and C. When one uses only the short description of QBS, then one could mistakenly believe that there are two different possibilities how to proceed resulting in two different sets of winners. **First**: When one starts with the set A and B, candidate B qualifies for election because he has a better Borda score than candidate A. Then one has to consider the set A and C; candidate A qualifies for election because he has a better Borda score than candidate C. As no seats remain to be filled, QBS terminates and the candidates A and B are the winners. **Second**: When one starts with the set A and C, candidate A qualifies for election because he has a better Borda score than candidate C. Then one has to consider the set A and B; however, as this set has already won one seat no additional candidate qualifies at stage 2. At stage 3, one observes that more than $v/(S+1)$ voters are solidly committed to the candidates A, B and D; however, as this set has already won one seat no additional candidate qualifies at stage 3. At stage 4, one observes that more than $2 \cdot v/(S+1)$ voters are solidly committed to the candidates A, B, C and D; as candidate D has the best Borda score candidate D qualifies for election. As no seats remain to be filled, QBS terminates and the candidates A and D are the winners.

Table 1: Solid Coalitions in Example 1

one candidate		two candidates		three candidates		four candidates	
candidate	No.	candidates	No.	candidates	No.	candidates	No.
A	30	A,B	34	A,B,C	7	A,B,C,D	71
B	17	A,C	37	A,B,D	34	A,B,C,E	
C	24	A,D		A,B,E		A,B,D,E	
D	29	A,E		A,C,D	30	A,C,D,E	
E		B,C		A,C,E		B,C,D,E	29
		B,D	29	A,D,E			
		B,E		B,C,D	29		
		C,D		B,C,E			
		C,E		B,D,E			
		D,E		C,D,E			
	100		100		100		100

However, the long description in "Voting Procedures" states clearly that when one has to decide how many additional seats a given solid coalition gets at a given stage then one has to consider as already qualified only those candidates who have already qualified at strictly earlier stages. In example 1, when one starts with the set A and C, candidate A qualifies for election because he has a better Borda score than candidate C. Then one has to consider the set A and B; as none of these candidates has already qualified at a strictly earlier stage, candidate B qualifies for election because he has a better Borda score than candidate A.

In short, to guarantee that the result doesn't depend on the order in which the solid coalitions are considered at a given stage, it is important that one looks only at those candidates who have qualified at *strictly* earlier stages. For example, suppose, at stage 10, one finds a set of 10 candidates such that more than $5 \cdot v/(S+1)$ voters, but not more than $6 \cdot v/(S+1)$ voters, are solidly committed to these 10 candidates. Suppose that already 4 of these 10 candidates have qualified at stages 1-9. Then that candidate of this set who has the best Borda score of all those candidates of this set who did not qualify at stages 1-9 qualifies at stage 10 *even if this set has already won additional seats at stage 10*.

At first sight, it isn't clear whether the QBS winners can be calculated in a polynomial runtime since there are 2^{C} possible sets of candidates. However, a set of candidates has to be taken into consideration only when at least one voter is committed to this set. In so far as at each of the **C** stages there cannot be more than v sets of candidates such that at least one voter is committed to this set, one has to take not more than $v \cdot$**C** sets of candidates into consideration to calculate the QBS winners. Therefore, a polynomial runtime is guaranteed.

When not each voter ranks all candidates, then Dummett's intention is met best when in each stage i those voters who don't strictly prefer all the candidates of some set of i candidates to every other candidate are allocated to no solid coalition.

Nicolaus Tideman writes about QBS [4]:

> To avoid sequential eliminations, Michael Dummett suggested a procedure in which a search would be made for solid coalitions of a size that deserved representation, and when such a coalition was found, an option (or options) that the coalition supported would be selected. If the solid coalition supported more than one option, the option (or options) with the greatest "preference score" (Borda count) would be selected. Preference scores would also be used to determine which options would fill any positions not filled by options supported by solid coalitions. I find Dummett's suggestion unsatisfying. Suppose there are voters who would be members of a solid coalition except that they included an "extraneous" option, which is quickly eliminated, among their top choices. These voters' nearly solid support for the coalition counts for nothing, which seems to me inappropriate.

At first sight, it isn't clear whether Tideman's criticism is feasible. It is imaginable that whenever there are "voters who would be members of a solid coalition except that they included an 'extraneous' option" there is also an STV method (i.e. a method where surpluses of elected candidates are transferred according to certain criteria to the next available preference and where, when seats remain to be filled, candidates are eliminated according to certain criteria and their votes are transferred to the next available preference) where this "nearly solid support for the coalition counts for nothing". If this is the case, then it is not appropriate to criticize QBS for ignoring this "nearly solid support". However, example 2 demonstrates that there are really situations where the QBS winners differ from the STV winners independently of the STV method used.

Example 2 (v = 100; S = 3; C = 5):

```
40 ACDBE.
39 BCDAE.
11 DABEC.
10 DBAEC.
```

The Borda scores are 252 for candidate A, 248 for candidate B, 237 for candidate C, 242 for candidate D, and 21 for candidate E. Table 2 lists all solid coalitions. At stage 1, the candidates A and B qualify for election because both candidates are preferred to every other candidate by more than $v/(S+1)$ voters each. At stage 2, it is observed that more than $v/(S+1)$ voters are solidly committed to the candidates A and C and that more than $v/(S+1)$ voters are solidly committed to the candidates B and C; but as both sets of candidates have already won one seat each, no additional candidate qualifies for election at stage 2. At stage 3, it is observed that more than $v/(S+1)$ voters are solidly committed to the candidates A, C, and D and that more than $v/(S+1)$ voters are solidly committed to the candidates B, C, and D; but as both sets of candidates have already won one seat each, no additional candidate qualifies for election at stage 3. At stage 4, it is observed that more than $3 \cdot v/(S+1)$ voters are solidly committed to the candidates A, B, C, and D; as this set has already won 2 seats, candidate D, the candidate with the best Borda score of all those candidates who haven't yet qualified, qualifies for election. As no seats remain to be filled, QBS terminates and the candidates A, B, and D are the winners. However, STV methods necessarily choose the candidates A, B, and C because, independently of how surpluses are transferred, candidate C always reaches the quota. In my opinion, example 2 questions whether compliance with proportionality for solid coalitions is sufficient for being a proportional preferential voting method.

Table 2: Solid Coalitions in Example 2

one candidate		two candidates		three candidates		four candidates	
candidate	No.	candidates	No.	candidates	No.	candidates	No.
A	40	A,B		A,B,C		A,B,C,D	79
B	39	A,C	40	A,B,D	21	A,B,C,E	
C		A,D	11	A,B,E		A,B,D,E	21
D	21	A,E		A,C,D	40	A,C,D,E	
E		B,C	39	A,C,E		B,C,D,E	
		B,D	10	A,D,E			
		B,E		B,C,D	39		
		C,D		B,C,E			
		C,E		B,D,E			
		D,E		C,D,E			
	100		100		100		100

Dummett's justification for his method is his claim that, unlike traditional STV methods, QBS is less "quasi-chaotic". He writes [3]:

> The defect of STV is that it is quasi-chaotic, in the sense that a small change in the preferences of just a few voters can have a great effect on the final outcome. This is because it may affect which candidate is eliminated at an early stage, and thus which votes are redistributed, this then affecting all subsequent stages of the assessment process.

However, in my opinion, example 3 demonstrates that also QBS is "quasi-chaotic". This is because a small change in the preferences can affect which candidate qualifies at an early stage, this then affecting all subsequent stages of the assessment process.

Example 3 (v = 100; S = 2; C = 5):

```
26 BCAED.
24 DCEBA.
10 EADBC.
 8 ABCED.
 7 EABDC.
 7 EDBCA.
 6 CDEBA.
 6 DEBCA.
 3 DCEAB.
 2 EBADC.
 1 DCBEA.
```

The Borda scores are 142 for candidate A, 216 for candidate B, 215 for candidate C, 204 for candidate D, and 223 for candidate E. Table 3 lists all solid coalitions. At stage 1, candidate D qualifies for election because more than $v/(S+1)$ voters strictly prefer candidate D to every other candidate. At stage 2, it is observed that more than $v/(S+1)$ voters are solidly committed to the candidates C and D; but as this set of candidates has already won one seat, no additional candidate of this set qualifies for election at stage 2. At stage 3, it is observed that more than $v/(S+1)$ voters are solidly committed to the candidates A, B, and C; as none of these candidates has already qualified, candidate B, the candidate with the best Borda score, qualifies for

Table 3: Solid Coalitions in the original Example 3

one candidate		two candidates		three candidates		four candidates	
candidate	No.	candidates	No.	candidates	No.	candidates	No.
A	8	A,B	8	A,B,C	34	A,B,C,D	
B	26	A,C		A,B,D		A,B,C,E	34
C	6	A,D		A,B,E	9	A,B,D,E	19
D	34	A,E	17	A,C,D		A,C,D,E	3
E	26	B,C	26	A,C,E		B,C,D,E	44
		B,D		A,D,E	10		
		B,E	2	B,C,D	1		
		C,D	34	B,C,E			
		C,E		B,D,E	13		
		D,E	13	C,D,E	33		
	100		100		100		100

election. As no seats remain to be filled, QBS terminates and the candidates B and D are the winners.

When a single DEBCA ballot is changed to BDECA, the Borda scores are 142 for candidate A, 218 for candidate B, 215 for candidate C, 203 for candidate D, and 222 for candidate E. Table 4 lists all solid coalitions for this modified example. At stage 1, no candidate qualifies for election. At stage 2, it is observed that more than $v/(S+1)$ voters are solidly committed to the candidates C and D; as candidate C has a better Borda score, candidate C qualifies for election. At stage 3, it is observed that more than $v/(S+1)$ voters are solidly committed to the candidates A, B, and C; but as this set of candidates has already won one seat, no additional candidate of this set qualifies for election at stage 3. At stage 4, it is observed that more than $v/(S+1)$ voters are solidly committed to the candidates A, B, C, and E and that more than $v/(S+1)$ voters are solidly committed to the candidates B, C, D, and E; but as both sets have already won one seat each, no additional candidates qualify for election at stage 4. At stage 5, candidate E qualifies for election because he has the best Borda score of all candidates who have not already qualified. Thus, by ranking candidate B higher candidate B is changed from a winner to a loser. By changing a single ballot the QBS winners are changed from the candidates B and D to the candidates C and E.

Table 4: Solid Coalitions in the modified Example 3

one candidate		two candidates		three candidates		four candidates	
candidate	No.	candidates	No.	candidates	No.	candidates	No.
A	8	A,B	8	A,B,C	34	A,B,C,D	
B	27	A,C		A,B,D		A,B,C,E	34
C	6	A,D		A,B,E	9	A,B,D,E	19
D	33	A,E	17	A,C,D		A,C,D,E	3
E	26	B,C	26	A,C,E		B,C,D,E	44
		B,D	1	A,D,E	10		
		B,E	2	B,C,D	1		
		C,D	34	B,C,E			
		C,E		B,D,E	13		
		D,E	12	C,D,E	33		
	100		100		100		100

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Sequential STV - a new version

I.D. Hill and Simon Gazeley

In Issue 2 of *Voting matters*, a system was reported called Sequential STV [1], designed to overcome, at least to some extent, the problem of premature exclusion of a candidate, which occurs when the one who has the fewest votes at the time is excluded, though due to receive many transfers later if only that exclusion had not taken place. That system has now been improved and we report here on the new version. One particular result of the improvement is that, in the case of a single seat, it is now certain to find the Condorcet winner if there is one.

The aim is to find a set of candidates of size *n*, where *n* is the number of seats to be filled, such that any set of $n+1$ candidates consisting of those *n* and 1 more, will result in the election of those *n* when an STV election is performed. When $n=1$ this reduces, of course, to the Condorcet rule. In a small election, or when $n=1$, it would be relatively easy and quick to do a complete analysis to find if there is such a set. The challenge is to find a way of doing so that will work in a reasonable time in large elections, where such a complete analysis would be impracticable. We recognise that the meanings of ‘a reasonable time’ and ‘impracticable’ are open to dispute, and that what is practicable will change as computers continue to get faster.

In the old version of Sequential STV, an initial STV count divided the candidates into probables and others, but the others were regarded as ‘in a heap’ and all of equal status. Consequently, if a challenger was successful, it would have been contrary to the axioms of anonymity and neutrality[2] to make a change of probables until all the others had been tested too, and that could lead to more than one challenger in the next main stage. In the new version the others are not put in a heap but in a queue, where the order depends upon the voting pattern. It is then fair to implement any change of probables at once, and the division of the method into main stages and sub-stages is no longer necessary.

How it works − the easy part

An initial STV count is made but instead of dividing into those elected and not elected, it classifies those who would have been elected as probables, and puts the others into a queue, in the reverse order of their exclusion in that initial count, except that the runner-up is moved to last place as it is already known that an initial challenge by that candidate will not succeed. Having found the probables and the order of the queue, further rounds each consist of $n+1$ candidates, the *n* probables plus the head of the queue as challenger, for the *n* seats.

It should be noted that, apart from the initial count, which is only to get things started, all counts are of *n*+1 candidates for *n* seats, so the ‘exclude the lowest’ rule, which is the least satisfactory feature of STV, is not used.

If the challenger is not successful, the probables are unchanged for the next round and the challenger moves to the end of the queue, but a successful challenger at once becomes a probable, while the beaten candidate is put to the end of the queue. The queue therefore changes its order as time goes on but its order always depends upon the votes.

The reordering of the queue during the count, by putting any losing candidate to the end of the queue, is to make sure that it cannot ever get into a state where, say, a set X are probables, A, B and C are all near the top of the queue and X+A beats X+B beats X+C beats X+A, while D is further down and X+D has not been tested. Putting losing candidates to the end means that D must head the queue at some point before A, B and C come round again.

This continues until either we get a complete run through the queue without any challenger succeeding, in which case we have a solution of the type that we are seeking, or we fall into a Condorcet-style loop. In the latter case, we have to enter the more difficult part, set out below, but it should be emphasised that in real elections, as distinct from specially devised test cases, that rarely happens.

How it works − the more difficult part

To decide that a loop has been found, a set that has been seen before must recur as the probables. If the queue is in the same order as before then a loop is certain and action must be taken at once. If, however, a set recurs but the queue is in a different order, it is conceivable though unlikely that something different, that breaks the loop, could happen. So, in that case, a second chance is given and the counting continues but, if the same set recurs yet again, a loop is assumed and action taken.

In either event the action is the same, to exclude all candidates who have never been a probable since the last restart (which means the start where no actual restart has occurred) and then restart from the beginning except that the existing probables and queue are retained instead of the initial STV count.

If there is no candidate who can be excluded, then a special procedure is used, in which any candidate who has always been a probable since the last restart is classified as a certainty and any other remaining candidate as a contender. From each possible set of *n*+1 candidates that includes all the certainties, an election for *n* seats is conducted. Since, at this point, most of the original candidates will be either excluded or certainties, there is no need to fear an astronomical number of tests needing to be made.

At the end of each test, the one candidate who has not reached the quota is assigned a fractional value calculated by dividing that candidate's votes by the quota. When all the tests have been done, the average of these fractions is calculated for each candidate. Additionally candidates are awarded one point for each contest in which they did reach the quota. It is these complete points that mainly decide, the average fraction being really only a tie-breaker.

The contender with the highest score is then reclassified as a certainty and, if the number of certainties is less than the number of seats, the special procedure is repeated with one contender fewer and one seat fewer to fill.

While this process may look complicated, it should be remembered that, on most occasions, only the part called ‘the easy part’ above is used, while the complications are used to sort out a Condorcet paradox if it occurs.

Programming

Where loops occur it will often be found that a particular set of candidates is being tested more than once. Storing results and accessing them as necessary would obviously be much quicker than repeating the same STV count many times. However, since most voting patterns do not have such loops, such storing of results would usually be unproductive extra work. For the present, the system has been programmed with repetition rather than storing.

The name ‘Sequential STV’

From now on the name Sequential STV will be used to mean this new version.

A random version

The initial STV count, to choose the initial probables and to determine the initial order of the queue, turns out to be not very important, in that an alternative version that selects the initial probables at random, and orders the initial queue at random, nearly always reaches the same eventual answer. It is fun to watch it getting from an initial nonsense selection to end up at the correct solution, but this version should not be used in practice because of rare cases where it can get a different result from that given by starting with an STV count and, where this is so, we suspect that it would usually be a less good result.

An example of such a rare case has been given previously[3] with a fictitious set of votes, having 4 candidates for 2 places, in which testing ABC elects AB and testing ABD elects AB, yet testing ACD elects CD and testing BCD elects CD. In that example, Sequential STV elects AB (which is, in fact, the better choice) whereas the random version has a 50-50 chance of finding either AB or CD. Such an example seems unlikely ever to occur in reality but the fact that it is possible means that it is better to guard against it by not using the random version.

Examples

With 5 candidates for 2 seats, consider the voting pattern

```
104 ABCD
103 BCDA
102 CDBA
101 DBCA
  3 EABCD
  3 EBCDA
  3 ECDBA
  3 EDCBA
```

Plain STV elects BC. Sequential STV chooses BC as probables, then tests BCD, BCE and BCA in that order. BC win each time and are elected.

Suppose, however, that the voters for A, B, C and D had all put in E as second preference to give (the example used in reference 1).

```
104 AEBCD
103 BECDA
102 CEDBA
101 DEBCA
  3 EABCD
  3 EBCDA
  3 ECDBA
  3 EDCBA
```

This evidently makes E a very much stronger candidate, for if any one of A, B, C or D had not stood, E would have been the first elected, but plain STV takes no notice, electing BC just as before. Sequential STV chooses BC as probables but then tests BCD, where BC stay as probables and D goes to the end of the queue, followed by BCE where BE become the new probables and C goes to the end of the queue. It then tests BEA and BED, BE winning each time. There is no need to test BEC again as that result is already known, so BE are elected.

Real voting patterns

In 43 real elections held on file, the sequential method merely confirmed the original result in 38 of them, and replaced just 1 candidate in 3 more of them. In only 2 cases were loops found, making it necessary to do more than the easy part of the method.

Timings

Some timings were made on an 11-year old PC with a 386 chip. In a real election with 10 candidates for 6 seats and 841 voters, simple STV took 11 seconds. Sequential STV made no change in those elected and took 23 seconds.

In a much more difficult case with 30 candidates for 15 seats and 563 voters, simple STV took 1 minute 6 seconds. Sequential STV found 1 candidate to be definitely replaced and 3 others who were in a loop for the final seat. It took a total of 18 minutes 30 seconds.

Should it be used?

With this new version, should it be recommended for practical use? That depends upon whether the user is willing to abandon the principle that it should be impossible for a voter to upset earlier preferences by using later preferences. Many people regard that principle as very important, but reducing the frequency of premature exclusions is important too. We know that it is impossible to devise a perfect scheme, and it is all a question of which faults are the most important to avoid.

In considering this, we need to take into account, among other things, that the true aim of an election should not be solely to match seats as well as possible to votes, but to match seats to the voters' wishes. Since we do not know the wishes we must use the votes as a substitute, but that makes it essential that the votes should match the wishes as far as possible. That, in turn, makes it desirable that the voters should not be tempted to vote tactically.

They would not be so tempted if they felt confident that later preferences were as likely to help earlier ones as to harm them, and if they could not predict the effect one way or the other. At present, we see no reason to doubt that these requirements are met.

All things considered, we believe that Sequential STV is worthy of serious consideration.

Comparison with STV(EES) and with CPO-STV?

STV(EES) [4] was designed to meet much the same aims as Sequential STV, and also has the same disadvantage that later preferences can upset earlier ones. A comparison of the two would be interesting. As at present defined, however, STV(EES) is so slow that a comparison is not easy. For an electoral method to be slow should not be considered too much of a disadvantage for real elections if it can be shown to get better results, but it is certainly a disadvantage for research purposes where a large number of counts of different data may be required within a reasonable time.

Using the examples above, STV(EES) elects BC from the first but BE from the second, just as Sequential STV does.

In the example given in section 6 of reference 4, AC were elected by STV(EES), which was not wrong as there was a paradox in the votes, but the paper admitted that 'I would still have preferred AB to be the winning set in this case', so it may be worth noting that Sequential STV does indeed elect AB.

CPO-STV [5, 6] was designed to search for an outcome that is globally optimum rather than merely locally stable. Again a comparison would be interesting.

Acknowledgements

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